ON THE STABILITY OF THE POISEUILLE FLOW IN A FLAT CHANNEL

(OB USTOICHIVOSTI TECHENIIA PUAZEILIA V PLOSKOM KANALE)

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The investigation of stability of the Poiseuille flow in a flat channel with respect to infinitely small perturbations is, as is known, reduced to solving the following problem. It is to determine whether for Equation (0.1)

$$(D^2-\alpha^2)^2 \varphi = i\alpha R \{(u-c)(D^2-\alpha^2)-u''\}\varphi \qquad \left(D=\frac{d}{dy}, \quad -1<y<1, \ u=1-y^2\right)$$

with boundary conditions

$$\varphi = D\varphi = 0 \quad \text{for} \quad y=\pm 1 \tag{0.2}$$

there exists an eigenvalue $c = c(\alpha, R)$ contained in the upper half-plane (for any values of positive parameters α and R). If such a $c = c_r + ic_i$, $c_i > 0$ does exist, then the Poiseuille flow is unstable.

This problem has attracted the attention of many investigators who had used analytical and numerical methods, and had arrived at different conclusions as regards stability (see historical review in [1]. Heisenberg was the first to deduce in 1924 [2] the Poiseuille flow instability. His conclusions were disputed for a long time, as it seemed paradoxical that viscosity phenomena could contribute to instability, and also because his mathematical analysis needed substantiation.

Proof is given in the following that the direction indicated by Heisenberg, Tollmien, Lin and Thomas [2 to 5] leads to the correct answer to the stability question. For convenience all references are made to the comprehensive review [1] and, therefore, do not indicate the historical sequence of investigations. In Section 1 a rigorous mathematical formulation of this problem is given. Section 2 deals with the analysis of the fundamental system of Equations (0.1). Finally, in Section 3 the characteristic determinant of problem (0.1),(0.2) is analysed, and the approximate eigenvalue of c is found.

1. The problem (0.1),(0.2) is considered, in accordance with [1], in a complex y-domain. The position of point y_c has clearly an important bearing on the analysis. It is defined as the root of Equation

$$u(y_c) - c = 0$$

it is the point at which the "degenerate" equation

$$(u-c)(D^2-\alpha^2)\varphi - u''\varphi = 0$$

derived from (0.1) for $\alpha R \to \infty$ has a singularity.

There are two such points for any $c \neq 1$. The analysis will be made with small values of c, so that points y_{1c} and y_{2c} will be close to -1 and $+1$, respectively

$$y_{1c} = -(1-c)^{1/2} = y_c, \quad y_{2c} = (1-c)^{1/2} = -y_c$$

(Here and in the following text the root of numbers positive and not tending to zero is understood as the approximation to the mean value). A linear substitution of the unknown c in the complex domain

$$z = \frac{2}{u'(y_c)}(y - y_c) = (1-c)^{-1/2}(y - y_c)$$

transforms problem (0.1),(0.2) into the following:

$$(D^2 - \beta^2)^2 \varphi = i\rho^2 \left\{ z\left(1 - \tfrac{1}{2} z\right)(D^2 - \beta^2) + 1 \right\} \varphi \qquad \left(D = \frac{d}{dz}\right) \tag{1.1}$$

$$\varphi = D\varphi = 0 \quad \text{for } z = z_1, \quad z = z_2 = 2 - z_1 \tag{1.2}$$

We have to find for Equation (1.1) a point $z = z_1$ at which condition (1.2) is fulfilled; with this, the complex parameters β^2 and ρ^2 will be expressed by

$$\beta^2 = \alpha^2 (1-c), \qquad \rho^2 = 2\alpha R (1-c)^2 \tag{1.3}$$

$$z_1 = z_1(c) = 1 - (1-c)^{-1/2} = -{}^1/_2 c\,(1 + O(c))$$
$$z_2 = z_2(c) = 2 - z_1 = 1 + (1-c)^{-1/2} = 2 + {}^1/_2 c\,(1 + O(c)) \tag{1.4}$$

Such a formulation of the problem has certain methodical advantages over the initial one. Instability corresponds to z_1 in the lower half-plane. As problem (0.1),(0.2) is an "even" one, so (1.1),(1.2) is also even with respect to point $z = 1$. Problem (0.1),(1.2) is analyzed by the asymptotic method for small α and large αR. At the beginning it can be limited to the problem of finding small eigenvalues of c, as evidently such c do exist. Stating the problem in form (1.1),(1.2) makes the following assumptions possible: because c is small, the dependence of β^2 and ρ^2 on c is probably insignificant. Therefore, an attempt at finding the first approximation can be made thus: substitute β^2 and ρ^2 for α^2 and αR as independent variables, find points z_1 and $z_2 = 2 - z_1$ at which the solution of $\varphi(z;\beta,\rho)$ together with the first derivative becomes zero, and compute c from Formula (1.4) using the derived value of z_1.

These considerations lead to the following.

1. An analysis is made of a certain fundamental system of solutions of Equation (1.1) consisting of a pair of "smooth" solutions which for $\rho \to \infty$ are close to the solution of the degenerate equation, and of a pair of the boundary value type solutions. The dependence of these solutions on β^2 and $\lambda = -i\rho^2$, considered as independent parameters, is analysed. Because coefficients of (1.1) are even, with respect to point $z = 1$, the fundamental system can be assumed to consist of even and odd functions. It appears that there exist "smooth" and boundary value type even solutions for which the fulfilment of (1.2) at point z_1 brings its fulfilment at point z_2.

2. The characteristic determinant of this pair of functions

$$f(z;\beta,\lambda) = \begin{vmatrix} \Phi(z;\beta,\lambda) & \psi(z;\beta,\lambda) \\ \Phi'(z;\beta,\lambda) & \psi'(z;\beta,\lambda) \end{vmatrix} \tag{1.5}$$

is analyzed in the neighborhood of the coordinate system origin $z = 0$ (because of the assumed smallness of c), and the approximate root $z_0(\beta,\lambda)$ of Equation $f(z;\beta,\lambda) = 0$ is derived.

3. The error of the root determination is evaluated, or more precisely, it is demonstrated that

$$[|z_0(\beta,\lambda) - Z(\beta,\lambda)| \ll |\operatorname{Im} z_0(\beta,\lambda)| \tag{1.6}$$

where $Z(\beta,\lambda)$ is the exact root of $f(z;\beta,\lambda) = 0$.

Thus, the sign of the true root imaginary part coincides with that of the imaginary part of the approximate root.

4. Finally, an analysis is made of Equation

$$z = Z(\beta(\alpha, z), \lambda(\alpha R, z))$$

where the dependence of β and λ on z is obtained substituting $c(z)$ from (1.4) for c. Proof is given that the solution of this equation is in the same area of the z-plane (i.e. solution for $\operatorname{Im} z < 0$). The proof of this statement is of an elementary topological character.

Unless otherwise stated, the analysis will be carried out in the domain of parameter variation as follows:

$$|\rho| \to \infty, \qquad |\arg \rho| \leqslant \varepsilon \tag{1.7}$$

$$|\beta| \to 0, \qquad |\arg \beta| \leqslant \varepsilon \tag{1.8}$$

$$|\rho|^{-2/3} \ll |z| \ll 1, \qquad |\operatorname{agr} z - \pi k| < \varepsilon \tag{1.9}$$

$$|\beta|^{-3} \ll |\rho| \ll |\beta|^{-5} \tag{1.10}$$

Here ε is a fixed small number (for example, $\varepsilon \leq {}^1/_{12}\pi$).

The reason for these limitations is the desire to ensure the applicability of asymptotic formulas, while their justification lies in that with the stated assumptions it is possible to prove the existence of instability. Limitations (1.9) and (1.10) can be weakened without much trouble; moreover, this becomes necessary for the analysis of the asymptotic behavior of the neutral curve (i.e. the curve in the αR-plane in which $c_i = 0$) expressed by $\beta^3 \rho \sim \text{const}$, $\beta^5 \rho \sim \text{const}$.

2. Equation (1.1) belongs to the class of equations analyzed by Wason [6]. In the formulation of its results, first the fundamental system of solutions of the degenerate nonviscous Equation is written

$$\{z(1 - {}^1/_2 z)(D^2 - \beta^2) + 1\}\varphi = 0 \tag{2.1}$$

which is obtained from (1.1) by a formal transition to limit at $\rho \to \infty$. Equation (2.1) is of the Fuchs type (see, for example, [8]) having characteristic values of zero and unity at singular points $z = 0$ and $z = 2$. It has, therefore, a fundamental system in the neighborhood of $z = 0$, consisting of a regular solution at zero

$$\varphi_1(z, \beta) = z \sum_{k=0}^{\infty} a_k(\beta) z^k \tag{2.2}$$

and a singular solution

$$\varphi_2(z, \beta) = \varphi_1 \operatorname{Ln} z + \chi(z, \beta), \qquad \chi(z, \beta) = \sum_{k=0}^{\infty} b_k(\beta) z^k \tag{2.3}$$

Series (2.2) and (2.3) are convergent for $|z| < 2$. Coefficients $a_k(\beta)$ and $b_k(\beta)$ are complete functions of β^2

$$a_k(\beta) = \sum_{l=0}^{\infty} a_{kl}\beta^{2l}, \qquad b_k(\beta) = \sum_{l=0}^{\infty} b_{kl}\beta^{2l} \tag{2.4}$$

Coefficients a_{kl} and b_{kl} are to be found from the recurrent relationships

$$a_{k+2,l+1} = \frac{1}{2}\left(1 - \frac{2}{k+1}\right) a_{k+1,l+1} + \frac{1}{(k+1)(k+2)}\left(a_{k,l} - \frac{1}{2} a_{k-1,l}\right) \tag{2.5}$$

$$a_{k+2,\,l+2} = \frac{1}{2}\left(1-\frac{2}{k+1}\right)b_{k+1,\,l+2} + \frac{1}{(k+1)(k+2)}\left(b_{k,l+1}-\frac{1}{2}\,b_{k-1,\,l+1}\right) - \\ - \frac{1}{(k+1)(k+2)}\left(a_{k+2,\,l+2}+a_{k+1,\,l+2}\right) - \frac{2}{k+2}\left(a_{k+2,\,l+2}-a_{k+1,\,l+2}\right) \qquad (2.6)$$

Functions $\varphi_1(z,\beta)$ and $\varphi_2(z,\beta)$ are determined with an approximation of the order of the constants.

In particular, assuming $a_{1,0}=1,\ a_{2,0}=-1/2,\ b_{1,1}=0,\ b_{2,1}=-1/2$, we obtain

$$\varphi_1(z,\beta) = z\{(1-1/2z)+\beta^2(1/6z^2+\ldots)+\ldots+\beta^{2n}(a_n z^{2n}+\ldots)+\ldots\}$$

$$\chi(z,\beta) = -1+z^2+\ldots+\beta^2(1/2 z^2+\ldots)+\ldots+\beta^{2n}(b_n z^{2n}+\ldots)+\ldots$$

This presentation of solutions, usually resorted to in the theory of differential equation analysis, has certain advantages over that of the Heisenberg series, often used in the theory of hydrodynamic stability, from the computation and theoretical points of view (cf. [1], Chapter III). Solution $\varphi_2(z,\beta)$ will obviously be a multiply-valued function in the neighborhood of $z=0$. In the following it will be necessary to separate the single-valued branch $\ln z$ from the multiply-valued function $\operatorname{Ln} z$.

For reasons explained below, the following branch will be considered

$$\ln z = \ln|z| + i\arg z, \qquad 1/2\pi \leqslant \arg z < -3/2\,\pi$$

(i.e. plane z is cut along ray $\operatorname{Re} z=0$, $\operatorname{Im} z>0$). In the following the analysis will be confined to the cut plane, and $\varphi_2(z,\beta)$ will be taken as the solution relative to $\ln z$, as selected above. Wason's conclusions [6] with respect to the fundamental system of solutions of Equation (1.1) will be formulated as follows.

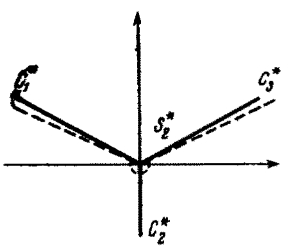

Fig. 1

Equation (1.1) will be considered in conjunction with Wason's model equation.

$$y^{(4)} + \lambda^2\{xy''+y\} = 0 \qquad (2.7)$$

(Here λ is a large parameter $|\lambda|\to\infty$.

The fundamental system of solutions of Equation (2.7) was the subject of a detailed analysis in [6a]. We shall describe it. Let S^* be a circle in the complex x-plane, defined by $|x|\leq a$. We introduce the following variable

$$\xi = {}^2/_3\,\lambda\,(-x)^{3/2}$$

and let C_k^* be three rays in the x-plane, defined by equations $\operatorname{Re}\xi=0$, i.e.

$$\arg x = {}^2/_3 k\pi - {}^2/_3\arg\lambda \qquad (k=1,\,2,\,3)$$

(The diagram relates to the case of $\arg\lambda = -\frac{1}{4}\pi$).

We denote by S_k^* an open sector limited by rays C_j^* $(j\neq k)$. For convenience of analysis of the asymptotic behavior of these solutions at $\lambda\to\infty$ we shall consider separately two cases:

$$1. \quad |\xi| \geqslant \xi_0 > 0, \qquad 2. \quad |\xi| \leqslant \xi_0$$

i.e. the asymptotic behavior in an arbitrarily fixed ξ_0-neighborhood of zero of the ξ-plane is different from the asymptotic behavior outside that neighborhood. In the x-plane this neighborhood contracts fairly rapidly to zero for $\lambda \to \infty$, i.e. this limitation is considerably weaker than, for example, that imposed by the fixed neighborhood configuration in the x-plane. It turns out that within $S_{\xi_0}{}^*$ (i.e. in the area $\xi_0|\lambda|^{-2/3} \leqslant x \leqslant a$) there exist solutions with the following asymptotic behavior:

(1) $D^mV(x, \lambda) = D^m[2\pi i x^{1/2}J_1(2x^{1/2})] + O(\lambda^{-2}) \qquad (m = 0, 1, 2, 3)$

this asymptotic character is true even within the full circle S^* ; this is a smooth regular solution.

(2) $D^mU_k(x, \lambda) = D^m[2\pi i x^{1/2}H_1^{(1)}(2x^{1/2})] + O(\xi^{-2}x^{1-m}) \qquad (m = 0, 1, 2, 3)$

this is a "smooth" singular solution in the closed sub-area $S - S_k{}^*$ with singularity at small $|x|$, $k = 1, 2, 3$.

(3) $D^mA_k(x, \lambda) = D^m[{}^3/_2\xi)^{-5/6}e^{\xi}](1 + O(\xi^{-1}) \qquad (m = 0, 1, 2, 3)$

in the closed sub-area $S^* - S_k^*$. Here one must assume that within S_k^* we have $\mathrm{Re}\,\xi < 0$. These solutions are of the boundary value type.

If we now consider a closed sub-sector S^* within area $S_{13}^* = S^* - S_2^*$, we find that there exists the following fundamental system of solutions (the independence of solutions is evident):

1. $V(x, \lambda)$ – a smooth regular solution
2. $U_2(x, \lambda)$ – a smooth singular solution
3. $A_1(x, \lambda)$ – a boundary value type solution decreasing from left to right (on axis $\mathrm{Im}\, x = 0$), with a negative exponent
4. $A_3(x, \lambda)$ – a boundary value type solution decreasing from right to left, with a positive exponent

A similar situation exists, as proved by Wason, in the case of the general equation of the form

$$y^{(4)} + \sum_{j=1}^{4} a_j(x)\, y^{(4-j)} + \lambda^2 \sum_{j=0}^{2} b_j(x)\, y^{2-i} = 0$$

with certain assumptions satisfied by Equation (1.1). More precisely, it is necessary to consider circle S

$$|z| < 2 - \delta$$

and the curvilinear rays C_j defined by Equations

$$\mathrm{Re}\,\xi = 0 \qquad (j = 1, 2, 3) \qquad (\xi = {}^2/_3\lambda(\Phi(z))^{3/2})$$

$$\left(\Phi(z) = \left\{\frac{3}{2}\int_0^z\left[-\zeta\left(1 - \frac{1}{2}\zeta\right)\right]^{1/2} d\zeta\right\}^{\frac{2}{3}}, \qquad \Phi(0) = 0, \qquad \Phi'(0) = 1\right.$$

Rays C_j originated at point $z = 0$, with tangents $C_j{}^*$, define the curvilinear sectors S_j which are models of sectors $S_j{}^*$ in the representation $x = \Phi(z)$. Then in the model $S_{\xi_0}{}^{**}$ of the closed subsector $S_{\xi_0}{}^{**}$ we have a fundamental system of solutions of the form as follows:

$$1)\ D^m \varphi_1(z;\beta,\lambda) = D^m \varphi_1(z,\beta) + O(\lambda^{-2}) \tag{2.8}$$

($\varphi_1(z, \beta)$ is a regular solution of the degenerate equation (2.1))

$$2)\ D^m\varphi_2(z;\beta,\lambda) = D^m\varphi_2(z,\beta) + R_m(z,\lambda) \tag{2.9}$$

$$R_0 = O(\lambda^{-2}z^{-2}),\quad R_1 = 0(\lambda^{-4/3}z^{-2}),\quad R_2 = O(z^{-1}),\ R_3 = O(z^{-2}),\quad R_m = O(\lambda^{-2}) \tag{2.10}$$

(for $|z| \geq z_0$, where z_0 is a fixed number)
($\varphi_2(z, \beta)$ is a singular solution of the degenerate equation (2.1))

$$3)\ D^m\varphi_3(z;\beta,\lambda) = D^m\Big[\chi(z)\,\lambda^{1/6}\Big(\frac{3}{2}\,\xi\Big)^{-1/6} e^{\xi}\Big]^{(m)}(1 + O(\xi^{-1}) \tag{2.11}$$

$$(\operatorname{Re}\xi < 0 \quad \text{for} \quad z > 0;\ m = 0, 1, 2, 3)$$

(with exponent $\lambda(z)$ diminishing to the right, a smooth function which does not vanish)

$$4)\ D^m\varphi_4(z;\beta,\lambda) = D^m\Big[\chi(z)\,\lambda^{1/6}\Big(\frac{3}{2}\,\xi\Big)^{-1/6} e^{\xi}\Big]^{(m)}(1 + O(\xi^{-1})) \tag{2.12}$$

$$(\operatorname{Re}\xi > 0 \quad \text{for} \quad z > 0;\ m = 0, 1, 2, 3)$$

(with exponent $\chi(z)$ increasing to the right, a smooth function which does not vanish)

Function $\Phi(z)$ can be presented in the explicit form

$$\Phi(z) = \Big\{\frac{3}{2}\int_0^z\Big(-\zeta\Big(1-\frac{\zeta}{2}\Big)^{1/2} d\xi\Big\}^{2/3} = \text{const}\big((z-1)\sqrt{z(1-{}^1/_2z)} + {} + \sin^{-1}\sqrt{z(1-{}^1/_2z)}\big)$$

with a natural selection of branches. It will be seen from this formula, or directly from the integral, that the segment of the real axis $-\epsilon \leq z \leq 2-\delta$ in the z-plane becomes a certain sector $-\epsilon' \leq x \leq a$ of the real axis in the x-plane, and in particular that axis $\operatorname{Im} z = 0$, $\operatorname{Re} z \geq 0$ is entirely contained in sector S_1 (as it is not intersected by any of the C_k, and because for small $|z|$ we have $C_k \sim C_k{}^*$). This proves the validity of the derived formulas in the neighborhood of point $z = 1$.

We note once again the fundamental features of the separated fundamental system of solutions. It contains two "smooth" functions $\varphi_1(z;\beta,\lambda)$ and $\varphi_2(z;\beta,\lambda)$ which are approximate solutions of the degenerate "nonviscous" equation (2.1). However, $\varphi_2(z;\beta,\lambda)$ may be considered as a smooth function only in area $S^*_{\xi_0}$; the two boundary value type solutions φ_3 and φ_4, roughly speaking, represent in S^* the exponential functions with exponent $S^*_{\xi_0}$. One of these functions decreases from the left end inwards of $S^*_{\xi_0}$, and the second from the right end. Thus, if point z_1 at which we wish to establish boundary conditions $\varphi(z_1) = D\varphi(z_1) = 0$ lies within $S^*_{\xi_0}$, and because point $z = 1$, as stated above, is also in the space $S^*_{\xi_0}$, we may expect to find conditions which for simpler problems are called "regular degeneration" [10]. This assumption proves to be correct.

E v e n s o l u t i o n s. Having proved the existence of two linearly independent even solutions with respect to point $z = 1$, one smooth, the other of the boundary value type, it is possible to reduce the rank of the characteristic determinant from four to two. This simplifies the exposition without affecting the principle.

For the formulation of the even solution of the boundary value problem it will be necessary to consider the Cauchy problem for (1.1) with initial data at point $z = 1$.

$$\Psi^{(m)}(1;\beta,\lambda)=\varphi_3^{(m)}(1;\beta,\lambda)\quad(m=0,2)\qquad\Psi^{(l)}(1;\beta,\lambda)=0\quad(l=1,3)$$

Function $\Psi(z;\beta,\lambda)$ is defined by Formula

$$\Psi(z;\beta,\lambda)=[C_1]\varphi_3(1;\beta,\lambda)\varphi_1(z;\beta,\lambda)+[C_2]\varphi_3(1;\beta,\lambda)\varphi_2(z;\beta,\lambda)+ + [1]\varphi_3(z;\beta,\lambda)+[\varphi_3^2(1;\beta,\lambda)]\varphi_4(z;\beta,\lambda) \tag{2.13}$$

$$C_j=\text{const},\ [a]\equiv a(1+O(\lambda^{-1}))$$

For small z solution $\Psi(z;\beta,\lambda)$ together with its derivatives has the same asymptotic character as $\varphi_3(z;\beta,\lambda)$, but is an even function with respect to $z=1$.

Of course, it is not possible to derive a smoth even solution by solving directly Cauchy's problem at point $z=0$, as inevitably the boundary layer will "grow" at the right- and left-hand side boundaries. There exists, however, an even boundary value function $\Psi(z;\beta,\lambda)$ which can be subtracted with suitable coefficients, leaving a smooth function. With this in mind, we shall solve Cauchy's problem for initial conditions as follows:

$$\Phi_1^{(m)}(1;\beta,\lambda)=\Phi^{(m)}(1,\beta)\quad(m=0,2);\qquad\Phi_1^{(l)}(1;\beta,\lambda)=0\quad(l=1,3)$$

where $\Phi(z,\beta)$ is the even solution of the "nonviscous" equation [6] expressed by

$$\Phi(z,\beta)=\varphi_1(z,\beta)+k\varphi_2(z,\beta) \tag{2 14}$$

(coefficients $k=k(\beta)$ will be derived below), then

$$\Phi_1(z;\beta,\lambda)=[1]\varphi_1(z;\beta,\lambda)+[k]\varphi_2(z;\beta,\lambda)+ + O(\lambda^{-4})\varphi_3(1;\beta,\lambda)\varphi_4(z;\beta,\lambda)+O(\lambda^{-4})\varphi_4(1;\beta,\lambda)\varphi_3(z;\beta,\lambda)$$

The last term of $\Phi_1(z;\beta,\lambda)$ contains factor $\varphi_4(1;\beta,\lambda)$ which is exponentially large. It is in fact the parasitically grown boundary value layer. It will disappear, however, if we subtract from $\Phi_1(z;\beta,\lambda)$ the even boundary value solution (2.13) after having it multiplied by the coefficient $O(\lambda^{-4})\varphi_4(1;\beta,\lambda)$ for $\varphi_3(z;\beta,\lambda)$ we obtain the sought smooth even solution

$$\Phi(z;\beta,\lambda)=[1]\varphi_1(z;\beta,\lambda)+[k]\varphi_2(z;\beta,\lambda)+O(\lambda^{-4})\varphi_3(1;\beta,\lambda)\varphi_4(z;\beta,\lambda) \tag{2.15}$$

Coefficients of φ_1 and φ_2 become terms of the order of $O(\lambda^{-4})$.

Before proceeding further it is necessary to analyse the root of equation close to the coordinate origin

$$f(z;\beta,\lambda)=\begin{vmatrix}\Phi(z;\beta,\lambda) & \Psi(z;\beta,\lambda)\\ \Phi'(z;\beta,\lambda) & \Psi'(z;\beta,\lambda)\end{vmatrix} \tag{2.16}$$

The dependence of the boundary value solution $\Psi(z;\beta,\lambda)$ on β will not matter; in this case it will be sufficient to consider the asymptotic term which does not contain β (as was done before). The dependence of function $\Phi(z;\beta,\lambda)$ on β is, on the other hand, very considerable.

We shall consider the even solutions of the nonviscous and of the degenerate characteristic equations. Coefficient $k(\beta)$ of the even solution $\Phi(z,\beta)$ of Equation (2.1) in Formula (2.14) can be presented in the form of a series

$$k(\beta)=\sum_{j=1}^{\infty}k_j\beta^{2j}\qquad\left(k(\beta)=-\frac{\varphi_1'(1,\beta)}{\varphi_2'(1,\beta)}\right)$$

Formulas for coefficient k_j are obtained from the basic equation of (2.14) (the numerator and denominator are power series with respect to β^2). These coefficients may also be found in the form of definite repetitive integrals of known functions, in particular

$$k_1=-2\int_0^1\left[z\left(1-\frac{1}{2}z\right)\right]^2dz<0$$

For our purposes it will be sufficient to know that

$$k(\beta) = k_1\beta^2 + O(\beta^4), \qquad k_1 < 0 \tag{2.17}$$

As the dependence of terms $\Phi(z;\beta,\lambda)$ and $\Phi'(z;\beta,\lambda)$ on λ in Equation (2.16) is weak and $\Psi'(z;\beta,\lambda)$ is $|\lambda|$-times greater than $\Psi(z;\beta,\lambda)$, it will be natural to take as the first approximation of z the root of Equation $\Phi(z;\beta,\lambda) = 0$, i.e. to equate to zero coefficient of the large term $\Psi'(z;\beta,\lambda)$ (of the order of λ). In view of the shown weak dependence of $\Phi(z;\beta,\lambda)$ on λ, it would again be natural to expect that the principal asymptotic term of this root will coincide with the principal asymptotic term of the root of the simpler equation $\Phi(z,\beta) = 0$ of the nonviscous characteristic equation.

For small z and β the following equality is true

$$\Phi(z,\beta) = z + O(z^2) + O(\beta^2 z) - k_1\beta^2 + O(\beta^2 z \ln z) = z - k_1\beta^2 + o(z) + o(\beta^2)$$

From this we obtain

$$z = k_1\beta^2 + o(\beta^2) \tag{2.18}$$

It will be shown in the following that the first term of (2.18) is in fact the principal asymptotic term of the root of Equation (2.16). The possibility of its derivation from the solution of the nonviscous problem had evidently not been realized earlier ([1], Chapter III).

3. Further analysis will be carried out as follows. The principal asymptotic terms of function $f(z;\beta,\lambda)$ are written down for large λ, and small β and z. These terms are denoted by $f_1(z;\beta,\lambda)$, and the approximate solution $z = z_0(\beta,\lambda)$ of Equation $f_1(z;\beta,\lambda) = 0$ is found. Next, we evaluate the difference $|Z(\beta,\lambda) - z_0(\beta,\lambda)|$ between solution $Z(\beta,\lambda)$ of Equation (2.16) and $z_0(\beta,\lambda)$. This difference is found to be smaller than the imaginary part of $z_0(\beta,\lambda)$ which means that $Z(\beta,\lambda)$ is in the same half-plane as $z_0(\beta,\lambda)$. Finally we make the following substitution:

$$\lambda = (2i\alpha R(1-c)^{1/3}, \qquad \beta^2 = \alpha^2(1-c)$$
$$c = c(z) = 1 - (1-z)^{-2} = (-2z + z^2)(1-z)^{-2}$$

and consider Equation

$$z = Z(\beta(z),\ \lambda(z))$$

It is shown that for large αR and small α^2 this equation has its solution $z(\beta,\lambda)$ close to solution $Z(\beta = \alpha,\ \lambda = (2i\alpha R)^{1/3})$, and in any case is in the same plane. Completion of this exercise provides the complete solution of the Poiseuille flow stability problem.

The even solutions (2.13) and (2.15), expressed in terms of the "nonviscous" solutions $\varphi_1(z;\beta)$ and $\varphi_2(z;\beta)$, and of the boundary value equation $\varphi_3(z;\beta,\lambda)$ decreasing from left to right, have the form

$$\Phi^{(m)}(z;\beta,\lambda) = \varphi_1^{(m)}(z,\beta) + k(\beta)\varphi_2^{(m)}(z,\beta) + O(\beta^2 R_m) =$$
$$= \Phi^{(m)}(z,\beta) + O(\beta^2 R_m) \qquad (m = 0, 1, 2, 3) \tag{3.1}$$
$$\Psi^{(m)}(z,\beta,\lambda) = \varphi_{31}^{(m)}(z;\beta,\lambda) + O(\lambda^{-P}) \quad (P \text{ is arbitrary})$$

(these estimates are made on the assumption that $\xi_0|\lambda|^{-2/3} \leqslant |z| \leqslant 2 - \delta$, $0 < \delta < 2$, $\xi_0 > 0$ are fixed). After substitution into (2.16), we have

$$\begin{vmatrix} \Phi(z,\beta)+O(\lambda^{-2})+O(\beta^2R_0) & \varphi_3(z;\beta,\lambda)+O(\lambda^{-p}) \\ \Phi'(z,\beta)+O(\lambda^{-2})+O(\beta^2R_1) & \varphi_3'(z;\beta,\lambda)+O(\lambda^{-p}) \end{vmatrix} = 0 \tag{3.2}$$

As an approximation of Equation (3.2) we can take Equation

$$\frac{\Phi(z,\beta)}{\Phi'(z,\beta)} = \frac{\varphi_3(z;\beta,\lambda)}{\varphi_3'(z;\beta,\lambda)} \tag{3.3}$$

For the determination of the approximate root of this equation it will be sufficient (as will be shown later) to use approximations as follows:

$$\frac{\Phi(z,\beta)}{\Phi'(z,\beta)} \approx z - k_1\beta^2, \qquad \frac{\varphi_3(z;\beta,\lambda)}{\varphi_3'(z;\beta,\lambda)} \approx -\rho^{-1}(-z)^{-1/2}\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right) \tag{3.4}$$

On comparing the two formulas of (3.4), and considering (assuming) that $\arg\beta^2 \ll \arg z_0$ (this assumption is based on formulas expressing c in terms of z, and β^2 in terms of $\sigma(z)$), we assume (3.5)

$$\operatorname{Re} z_0 = \operatorname{Re}(k_1\beta^2), \qquad \operatorname{Im} z_0 = -\operatorname{Re}\frac{1}{\rho\sqrt{-z}}\frac{\sqrt{2}}{2} = -\frac{\sqrt{2}}{2}\sqrt{-k_1}\operatorname{Re}\frac{1}{\rho}\frac{1}{\beta} < 0$$

This defines a certain approximate root of Equation (2.16).

The error made in the determination of this root may be estimated in several ways. In order to avoid any arguments about the analytic nature of our equations (incidentally, not very complicated), we shall use here Newton's method in the form given to it by Kantorovich [8] in preference to the often used Runge theorem.

This method, in its application to the solution of equation $f(z) = 0$, can be stated as follows: let z_0 be the approximate root of Equation

$$f(z) = 0, \qquad \left|\frac{f(z_0)}{f'(z_0)}\right| \leqslant \eta, \qquad \max_z\left|\frac{f''(z)}{f'(z_0)}\right| \leqslant K$$

with $h = \eta$ and $K \to 0$ with respect to a certain parameter. Then the true root will be within the circle $|z - z_0| \leq \eta$ (condition that $K \to 0$ is not essential, it is used here to simplify the theorem formulation). As only the sign of the imaginary part $\operatorname{Im} z(\beta, \lambda)$ is important, it will be sufficient to prove that

$$\eta = O(\operatorname{Im} z_0) = O(\rho^{-1}\beta^{-1})$$

Recalling that (in accordance with (1.5))

$$f(z;\beta,\lambda) = \Phi(z;\beta,\lambda)\Psi'(z;\beta,\lambda) - \Phi'(z;\beta,\lambda)\Psi(z;\beta,\lambda)$$

and using estimates (2.8) to (2.12) and (2.17), we easily obtain

$$f(z_0;\beta,\lambda) = \varphi_3(z_0;\beta,\lambda)\,o(\beta^{4-\varepsilon})\,O(\beta\rho), f'(z;\beta,\lambda) = \varphi_3(z_0;\beta,\lambda)\,O(\beta\rho)$$
$$f''(z;\beta,\lambda) = \varphi_3(z_0;\beta,\lambda)\,O(\beta^2\rho^2)$$

Hence,

$$k_\eta = o(\beta^{4-\varepsilon})\,O(\beta\rho) \to 0, \qquad \eta = o(\beta^{4-\varepsilon}) = O(\rho^{-1}\beta^{-1}) \quad \text{for} \quad \rho = O(\rho^{-1/(5-\varepsilon)})$$

This proves that Equation (2.16) has a root $z = Z(\beta, \lambda)$ contained in circle

$$|z_0(\beta,\lambda) - z| \leqslant \eta = o(\rho^{-1}\beta^{-1}) \tag{3.6}$$

with the negative imaginary part of parameter

$$\operatorname{Im} z(\beta,\lambda) = -\tfrac{1}{2}\sqrt{2}\,(-k_1)^{1/2}\operatorname{Re}\rho^{-1}\beta^{-1}(1 + O(\rho^{-1}\beta^{-1})$$

However, it is the root $z(\alpha^2, \alpha R)$ of the following equation that has to be found

$$z = Z(\alpha^2(1 - c(z)), \qquad 2i\alpha R(1 - c(z)) = Z\left(\frac{\alpha^2}{(1-z)^2}, \frac{-2i\beta R}{(1-z)^2}\right)$$

Proof of the existence of this root is obtained by applying the principle of the stationary point to the representation of circle $|z_0 - z| \leqslant C|\beta^{-1}\rho^{-1}|$ by means of the Z-function. By virtue of Formulas (3.5) and (3.6) this circle penetrates inside the circle of radius $|z - z_0| = o(\rho^{-1}\beta^{-1})$.

Notes 1. It is easily seen that a more extensive application of results of Wason's work would make it possible to derive the asymptotic character of the neutral curve branches in the α, R-plane. Specifically, this curve has two branches defined by Equations $R \sim \alpha^{-7}$ and $R \sim \alpha^{-11}$. This result [1] can be proved as strictly, as the finding of points inside a curve.

2. The method applied here may also be used for the derivation of eigenvalues $c = c(\alpha, R)$, and in this sense it is simpler than the Heisenberg-Lin method [1].

3. The problem of boundary layer stability on a flat plate, and other problems concerning plane-parallel flow stability losses, may be analized by this method.

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